

HÖLDER ESTIMATES FOR CAUCHY-TYPE INTEGRALS AND PROPER HOLOMORPHIC MAPPINGS OF SYMMETRIC PRODUCTS

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ABSTRACT. We prove estimates in Hölder spaces for some Cauchy-type integral operators representing holomorphic functions in Cartesian and symmetric products of planar domains. As a consequence, we obtain information on the boundary regularity in Hölder spaces of proper holomorphic maps between symmetric products of planar domains.

1. INTRODUCTION

Let $\pi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be the proper holomorphic map (called the *symmetrization map*) given by

$$\pi(z) = (\pi_1(z), \dots, \pi_n(z)), \quad (1.1)$$

where $\pi_j(z)$ is the j -th elementary symmetric polynomial in n variables. Recall that $\pi_1(z) = \sum_k z_k$, $\pi_2(z) = \sum_{k < l} z_k z_l$, etc. If U is a bounded domain in the plane, we denote by $\Sigma^n U$, the n -fold *symmetric product* of the domain U with itself, which is by definition the image of the n -fold Cartesian product $U^n = U \times \dots \times U$ under the map π :

$$\Sigma^n U = \pi(U^n). \quad (1.2)$$

$\Sigma^n U$ is a pseudoconvex domain in \mathbb{C}^n with non-Lipschitz boundary (see [4, Proposition 5.3]). When $U = \mathbb{D}$, the unit disc, the domain $\Sigma^n \mathbb{D}$, under the name *symmetrized polydisc* arises naturally in problems of control theory (see [1]), and has been studied intensively, both from the operator-theoretic and function-theoretic point of view (see e.g. [2, 5, 6, 8, 9, 3, 4] etc). In particular, given two bounded planar domains U and V , the proper holomorphic maps from $\Sigma^n U$ to $\Sigma^n V$ have been classified: they turn out to be functorially induced by proper holomorphic maps from U to V (see [8, 9, 4], and Section 9 below).

It is natural to ask how these proper holomorphic maps behave at the boundary, and this question is especially interesting since the boundaries of $\Sigma^n U$ and $\Sigma^n V$ are non-Lipschitz. It was shown in [4] that if U and V have \mathcal{C}^∞ -smooth boundaries, then a proper holomorphic map from $\Sigma^n U$ to $\Sigma^n V$ extends to a \mathcal{C}^∞ map of the closures. In this paper, we consider the situation when ∂U and ∂V have finite order of differentiability. Let $\mathcal{C}^{k,\alpha}(U)$ denote the space of functions on a domain U whose k -th order partial derivatives exist and satisfy a uniform Hölder condition of order α . For a domain $U \subset \mathbb{C}^n$, we denote by $\mathcal{A}^{k,\alpha}(U)$ the space $\mathcal{O}(U) \cap \mathcal{C}^{k,\alpha}(U)$ of holomorphic functions or maps which are of class $\mathcal{C}^{k,\alpha}(U)$. Our main result in this direction is the following:

Theorem 1.1. *Let $n \geq 1, k \geq 0$ be integers, and let*

$$\Lambda_n = \begin{cases} n! & \text{if } n \leq 3 \\ \frac{3}{2}n! & \text{if } n > 3. \end{cases} \quad (1.3)$$

If U, V are planar domains with boundaries of class $\mathcal{C}^{n(k+1)+1}$ and $F : \Sigma^n U \rightarrow \Sigma^n V$ is a proper holomorphic map, then $F \in \mathcal{A}^{k, \frac{\theta}{\Lambda_n}}(\Sigma^n U)$ for each $0 < \theta < 1$.

Note the loss of derivatives – which is not surprising if one remembers that symmetric products have highly singular non-Lipschitz boundaries.

The proof of Theorem 1.1 can be reduced to the study of the regularity in Hölder spaces of certain integral operators closely related to the classical Cauchy operator. Let Γ be the boundary of a smoothly bounded planar domain U , and let $p : \mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic polynomial in $(n+1)$ complex variables. We can associate with p an integral operator \mathcal{D}_p defined by

$$\mathcal{D}_p \phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(t)}{p(z, t)} dt, \quad (1.4)$$

acting on continuous functions defined on Γ . Note that $\mathcal{D}_p \phi$ is a holomorphic function on the open set $\mathbb{C}^n \setminus \Gamma_p^*$, where

$$\Gamma_p^* = \{ w \in \mathbb{C}^n \mid \text{there is a } t \in \Gamma \text{ such that } p(w, t) = 0 \}. \quad (1.5)$$

When $n = 1$ and $p(z, t) = t - z$, we obtain the classical *Cauchy Transform* or the *Cauchy Integral*, the operator \mathcal{T} given by

$$\mathcal{T} \phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(t)}{t - z} dt, \quad (1.6)$$

which defines $\mathcal{T} \phi$ as a holomorphic function for $z \in \mathbb{C} \setminus \Gamma$. For our application, we will consider two special cases of the general integral \mathcal{D}_p . In our first example, we take $p = q_n$, the polynomial of degree n given by

$$q_n(z_1, \dots, z_n, t) = t^n - z_1 t^{n-1} + z_2 t^{n-2} - \dots + (-1)^n z_n. \quad (1.7)$$

For this q_n , if $\mathbb{C} \setminus \Gamma$ has κ components, then one can show that $\mathbb{C}^n \setminus \Gamma_{q_n}^*$ consists of $\binom{n+\kappa-1}{\kappa-1}$ connected components (see Section 3), one of which is precisely the symmetric product $\Sigma^n U$ defined in (1.2). We denote by \mathcal{E}_n the operator defined by restriction of \mathcal{D}_{q_n}

$$\mathcal{E}_n \phi = (\mathcal{D}_{q_n} \phi)|_{\Sigma^n U}, \quad (1.8)$$

which maps continuous functions on Γ to holomorphic functions on $\Sigma^n U$. Note that $\mathcal{E}_1 = \mathcal{T}$, the Cauchy transform, and we may call \mathcal{E}_n the *symmetrized Cauchy transform in n variables*. We prove the following regularity result for the map \mathcal{E}_n :

Theorem 1.2. *Let $k \geq 0, n \geq 1$ be integers and let the boundary Γ of U be of class $\mathcal{C}^{n(k+n+1)}$. Then, for $0 < \alpha < 1$, the map \mathcal{E}_n is continuous from the space $\mathcal{C}^{(k+1)n-1, \alpha}(\Gamma)$ to the space $\mathcal{A}^{k, \frac{\alpha}{\Lambda_n}}(\Sigma^n U)$, where Λ_n is as in (1.3)*

Here $\mathcal{C}^{k, \alpha}(\Gamma)$ is the space of functions on the smooth curve Γ which are k times continuously differentiable with respect to arc length, and such that the k -th derivative is Hölder continuous of order α . Our second example corresponds to the choice $p = \omega_n$, where

$$\omega_n(z_1, \dots, z_n, t) = \prod_{j=1}^n (t - z_j). \quad (1.9)$$

Assuming that $\mathbb{C} \setminus \Gamma$ has κ components, the space $\mathbb{C}^n \setminus \Gamma_{\omega_n}^*$ has κ^n components, and one of these components is the n -fold cartesian product U^n . We define the operator \mathcal{B}_n again by restriction to U^n :

$$\mathcal{B}_n \phi = (\mathcal{D}_{\omega_n} \phi)|_{U^n}, \quad (1.10)$$

Note that $\mathcal{B}_1 = \mathcal{T}$, and for $n \geq 2$, we will call \mathcal{B}_n the *Cauchy-Nørlund Transform* (cf. [16, p. 199]). Denote by $\mathcal{A}_{\text{sym}}^{k,\alpha}(U^n)$ the intersection $\mathcal{C}^{k,\alpha}(U^n) \cap \mathcal{O}_{\text{sym}}(U^n)$, where $\mathcal{O}_{\text{sym}}(U^n)$ is the space of *symmetric holomorphic functions*. Recall that a function $f \in \mathcal{O}(U^n)$ is *symmetric* if for each permutation σ in the symmetric group S_n and for each $z \in U^n$, we have $f(z) = f(\sigma(z))$, where

$$\sigma(z_1, \dots, z_n) = (z_{\sigma(1)}, \dots, z_{\sigma(n)}). \quad (1.11)$$

Our main result regarding \mathcal{B}_n is:

Theorem 1.3. *Let $k \geq 0, n \geq 1$ be integers and let the boundary Γ of U be of class \mathcal{C}^{k+n+1} . Then, for $0 < \alpha < 1$, the Cauchy-Nørlund Transform \mathcal{B}_n is a continuous linear map from $\mathcal{C}^{k+n-1,\alpha}(\Gamma)$ to $\mathcal{A}_{\text{sym}}^{k,\alpha}(U^n)$.*

This paper is organized as follows. After reviewing some elementary properties of the operators \mathcal{B}_n and \mathcal{E}_n introduced above in Section 3, in Section 4 we introduce the key observation (Proposition 4.1), that \mathcal{B}_n can be represented in terms of *divided differences*, discrete analogs of derivatives which occur in Newton's interpolation formula. This, along with a representation (4.4) of the divided difference allow us to prove Theorem 1.3 for convex domains U , and the general case follows in Section 6 by conformal mapping and a covering argument using the fact that bounded locally Hölder functions are Hölder. In Section 7 Theorem 1.2 is deduced from Theorem 1.3 and a distortion estimate for the map π , which we obtain using the results of [12]. Finally, in Section 9, a representation of proper holomorphic maps between symmetric products as an integral from [4] is used along with Theorem 1.2 to obtain Theorem 1.1. We note here that the statements of Theorem 1.2 and Theorem 1.1 can probably be improved, and slightly better regularity results may be obtained by more work. The non-optimality is because of possible loss of information in the course of the proof. It will be interesting to obtain sharp results in this direction.

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3. REMARKS ON THE OPERATORS \mathcal{B}_n AND \mathcal{E}_n

The integral (1.4) defining \mathcal{D}_p makes sense at $z \in \mathbb{C}^n$, provided $p(z, t)$ does not vanish for any $t \in \Gamma$, i.e., $z \in \mathbb{C}^n \setminus \Gamma_p^*$. By differentiation under the integral sign, $\mathcal{D}_p \phi$ is a holomorphic function on $\mathbb{C}^n \setminus \Gamma_p^*$, and by writing

$$\mathbb{C}^n \setminus \Gamma_p^* = \bigcap_{t \in \Gamma} \{ z \in \mathbb{C}^n \mid p(z, t) \neq 0 \},$$

we see that $\mathbb{C}^n \setminus \Gamma_p^*$, being the intersection of complements of complex hypersurfaces is pseudoconvex. In general, $\mathbb{C}^n \setminus \Gamma_p^*$ is not connected, and therefore, each of its components is pseudoconvex.

Suppose that $\mathbb{C} \setminus \Gamma$ consists of κ components, U_1, \dots, U_κ , where we can assume that $U_1 = U$ (where $\partial U = \Gamma$), U_2 is unbounded and the other $\kappa - 1$ components are bounded. When $p = q_n$, with q_n as in (1.7), we can associate with any κ -tuple of non-negative integers (m_1, \dots, m_κ) with $\sum_{j=1}^\kappa m_j = n$, the subset of $\mathbb{C}^n \setminus \Gamma_{q_n}^*$ given by

$$U(m_1, \dots, m_\kappa) = \{ z \in \mathbb{C}^n \mid m_j \text{ roots of } q_n(z, t) \text{ lie in } U_j, \text{ for } j = 1, \dots, \kappa \},$$

where for a fixed z , we think of $q_n(z, t)$ as a polynomial in the variable t . An argument using continuity of roots as functions of coefficients shows that each of $U(m_1, \dots, m_\kappa)$ is connected. Since $\mathbb{C}^n \setminus \Gamma_{q_n}^*$ is the disjoint union of the sets $U(m_1, \dots, m_\kappa)$ as the κ -tuple (m_1, \dots, m_κ) ranges over all combinations such that $\sum_{j=1}^\kappa m_j = n$, it follows that the connected components of the set $\mathbb{C}^n \setminus \Gamma_{q_n}^*$ are precisely the $U(m_1, \dots, m_\kappa)$, of which, therefore, there are a total of $\binom{n+\kappa-1}{\kappa-1}$. We are particularly interested in the component $U(n, 0, \dots, 0)$, which is described by

$$U(n, 0, \dots, 0) = \{ z \in \mathbb{C}^n \mid \text{all roots of } q_n(z, t) \text{ lie in } U \}.$$

We claim that $U(n, 0, \dots, 0) = \Sigma^n U = \pi(U^n)$. Indeed, let $z \in U(n, 0, \dots, 0)$, and let w_1, \dots, w_n be the roots of the polynomial $q_n(z, t) = t^n - z_1 t^{n-1} + \dots + (-1)^n z_n$. Then, by the definition of $U(n, 0, \dots, 0)$, each w_j belongs to U . Consider the point $w \in U^n$ given by $w = (w_1, \dots, w_n)$. Then, by the relationship between the roots and coefficients of a polynomial, we see that $z = \pi(w)$ so that $z \in \Sigma^n U$. Conversely, if $z \in \Sigma^n U$, let $w = (w_1, \dots, w_n) \in U^n$ be such that $z = \pi(w)$. Then clearly $w_1, \dots, w_n \in U$ are the roots of the equation $q_n(z, t) = 0$, so that $z \in U(n, 0, \dots, 0)$.

Note that the above argument also proves the following fact, which will be used later. If $z, w \in U^n$ are such that $\pi(z) = \pi(w)$, then there is a permutation $\sigma \in S_n$ such that $z = \sigma(w)$, where $\sigma(w)$ is as in (1.11). Analogously, note that when $p = \omega_n$,

$$\begin{aligned} \mathbb{C}^n \setminus \Gamma_{\omega_n}^* &= \{ z \in \mathbb{C}^n \mid t - z_j \neq 0 \text{ for all } t \in \Gamma \text{ and } j = 1, \dots, n \} \\ &= (\mathbb{C} \setminus \Gamma)^n. \end{aligned}$$

Therefore, when $\mathbb{C} \setminus \Gamma$ has κ components, $(\mathbb{C} \setminus \Gamma)^n$ has κ^n components. In defining the operator \mathcal{B}_n , we restrict our attention to the component U^n .

The restriction of \mathcal{B}_n to U_n and of \mathcal{E}_n to $\Sigma^n U$ is motivated by the application to proper maps of symmetric products as in Theorem 1.1. It is interesting to ask what happens in the other components, and whether there is a non-trivial analog of the Sokhotski-Plemelj jump formula for these transforms. Preliminary investigations show that we should not expect boundedness in Hölder topologies of the maps \mathcal{B}_n and \mathcal{E}_n when restricted to unbounded components of $\mathbb{C}^n \setminus \Gamma_p^*$. Further, along $\partial \Sigma^n U$ and ∂U^n , the $\mathcal{E}_n \phi$ and $\mathcal{B}_n \phi$ cannot be defined as Cauchy Principal values for Hölder continuous ϕ on Γ , and in fact the integral defining the Cauchy Principal value blows up at some points of the boundary, depending on the geometry near that point. If we define for $z \in \partial U^n$ the set $\Gamma(z, \rho) \subset \mathbb{C}$ by $\Gamma(z, \rho) = \bigcup_{j=1}^n B(z_j, \rho)$, where $B(z_j, \rho)$ the disc centered at z_j and radius ρ , a computation shows that if $\phi \in \mathcal{C}^\alpha(\Gamma)$ for some $0 < \alpha < 1$, then

$$\int_{\Gamma \setminus \Gamma(z, \rho)} \frac{\phi(t)}{\omega_n(z, t)} dt = O\left(\rho^{1-\chi(z)}\right),$$

where for a point $z \in \partial U^n$, we denote by $\chi(z)$ the largest integer k such that we can find k numbers in the n -tuple (z_1, \dots, z_n) which are equal to each other. In particular, this shows that there is no hope for a simple generalization of the jump formula to the operator \mathcal{B}_n .

4. A REPRESENTATION OF \mathcal{B}_n IN CONVEX DOMAINS

We now derive some representation formulas for the Cauchy-Nørlund transform, which are classical in numerical analysis ([16, 18, 7]). Let f be a holomorphic function on $\Omega \subset \mathbb{C}$. Recall that the n -th divided difference of f is a function $f^{[n]}$ of $n+1$ variables which occurs as a coefficient in the Newton Interpolation formula (see [7].) For our purposes, the divided difference $f^{[n]}$ of order n may be defined as follows: $f^{[0]}(z_1) = f(z_1)$, and we set

$$f^{[1]}(z_1, z_2) = \frac{f^{[0]}(z_1) - f^{[0]}(z_2)}{z_1 - z_2},$$

which is defined a priori if $z_1 \neq z_2$. But the numerator is a holomorphic function on Ω^2 which vanishes on the analytic variety $z_1 - z_2 = 0$, and therefore, the right hand side extends as a holomorphic function on Ω^2 . We recursively define $f^{[n]}$ by

$$f^{[n]}(z_1, z_2, \dots, z_{n+1}) = \frac{f^{[n-1]}(z_1, \dots, z_n) - f^{[n-1]}(z_2, \dots, z_{n+1})}{z_1 - z_n}. \quad (4.1)$$

It follows as above that $f^{[n]}$ extends to an element of $\mathcal{O}(\Omega^{n+1})$. We also denote by $\Delta^n : \mathcal{O}(\Omega) \rightarrow \mathcal{O}(\Omega^{n+1})$ the mapping $f \mapsto f^{[n]}$.

Proposition 4.1. *The map \mathcal{B}_n can be represented in terms of divided differences of the Cauchy transform \mathcal{T} of (1.6) as:*

$$\mathcal{B}_n = \Delta^{n-1} \circ \mathcal{T}. \quad (4.2)$$

Proof. For $n = 1$, both sides of (4.2) are equal (to the Cauchy transform.) Let ϕ be a continuous function on Γ and let $f = \mathcal{T}\phi$. Assuming that for a certain n , we have $\mathcal{B}_n\phi = f^{[n-1]}$, we compute $f^{[n]}$:

$$\begin{aligned} f^{[n]}(z_1, \dots, z_{n+1}) &= \frac{1}{z_1 - z_{n+1}} \left(f^{[n-1]}(z_1, \dots, z_n) - f^{[n-1]}(z_2, \dots, z_{n+1}) \right) \\ &= \frac{1}{2\pi i} \cdot \frac{1}{z_1 - z_{n+1}} \int_{\Gamma} \phi(t) \left(\frac{1}{(t - z_1) \dots (t - z_n)} - \frac{1}{(t - z_2) \dots (t - z_{n+1})} \right) dt \\ &= \frac{1}{2\pi i} \cdot \frac{1}{z_1 - z_{n+1}} \int_{\Gamma} \phi(t) \frac{(t - z_{n+1}) - (t - z_1)}{(t - z_1) \dots (t - z_{n+1})} dt \\ &= \mathcal{B}_{n+1}\phi(z_1, \dots, z_{n+1}). \end{aligned}$$

This continues to hold when $z_1 = z_{n+1}$ thanks to the fact that both sides are analytic. The result follows by induction. \square

As a consequence we obtain a property of divided differences which is far from obvious from the representation (4.1).

Corollary 4.2. *If $f \in \mathcal{O}(\Omega)$, where Ω is an open subset of \mathbb{C} , then $f^{[n-1]} \in \mathcal{O}_{\text{sym}}(\Omega^n)$, the space of symmetric holomorphic functions on the Cartesian power Ω^n .*

Proof. Given a point $(z_1, \dots, z_n) \in \mathbb{C}^n$, let $U \Subset \Omega$ be a smoothly bounded open set containing each of the points z_1, z_2, \dots, z_n and let Γ be the boundary of U . If we set $\phi = f|_{\Gamma}$, by the Cauchy integral theorem, $\mathcal{B}_1\phi = f$ on U , where the Cauchy integral $\mathcal{B}_1\phi$ is taken along Γ . But then, by Proposition 4.1, $f^{[n-1]} = \mathcal{B}_n\phi$. Since $\mathcal{B}_n\phi$ is obviously in $\mathcal{O}_{\text{sym}}(U^n)$, the result follows. \square

To state the next result, we recall some definitions. For an integer $d \geq 0$, we denote by Σ_d the standard d -simplex in \mathbb{R}^{d+1} :

$$\Sigma_d = \left\{ x \in \mathbb{R}^{d+1} \left| \sum_{j=1}^{d+1} x_j = 1, \text{ and } x_j \geq 0 \text{ for } j = 1, \dots, d+1 \right. \right\},$$

and let A_d be the d -simplex in \mathbb{R}^d given by

$$A_d = \left\{ x \in \mathbb{R}^d \left| \sum_{j=1}^d x_j \leq 1, \text{ and } x_j \geq 0 \text{ for } j = 1, \dots, d \right. \right\}.$$

Then Σ_d is the graph of the function $(x_1, \dots, x_d) \mapsto 1 - \sum_{j=1}^d x_j$ over A_d , i.e.,

$$\Sigma_d = \left\{ x \in \mathbb{R}^{d+1} \left| x_{d+1} = 1 - \sum_{j=1}^d x_j, \text{ and } (x_1, \dots, x_d) \in A_d \right. \right\}.$$

Therefore, on Σ_d , we can take (x_1, \dots, x_d) as coordinates, which we will refer to as the *standard coordinates* of Σ_d .

We denote the d -dimensional Hausdorff measure by \mathcal{H}_d . Note that since Σ_d is contained in an affine hyperplane of \mathbb{R}^{d+1} , the d -dimensional Hausdorff measure on Σ_d is simply the usual surface measure on the affine hyperplane. Using the fact that Σ_d is the graph of the function $u(x_1, \dots, x_d) = 1 - \sum_{j=1}^d x_j$, we can represent the Hausdorff measure on Σ_d in terms of the standard coordinates:

$$\begin{aligned} d\mathcal{H}_d &= \sqrt{1 + \sum_{j=1}^d \left(\frac{\partial u}{\partial x_j} \right)^2} dx_1 dx_2 \dots dx_d \\ &= \sqrt{1 + d} \cdot dx_1 dx_2 \dots dx_d. \end{aligned} \tag{4.3}$$

We denote by $\langle z, w \rangle$ the standard Hermitian inner product $\sum z_j \overline{w_j}$. We prove a complex version of a classical representation of divided differences:

Proposition 4.3 (Genocchi-Hermite formula). *Let U be a convex domain in \mathbb{C} and $f \in \mathcal{O}(U)$. Then, for $z \in U^{n+1}$:*

$$f^{[n]}(z) = \frac{1}{\sqrt{n+1}} \int_{\Sigma_n} f^{(n)}(\langle z, \tau \rangle) d\mathcal{H}_n(\tau). \tag{4.4}$$

Proof. We proceed by induction. When $n = 0$, the left hand side is $f^{[0]}(z) = f(z)$, $\langle z, \tau \rangle = z$, and the right hand side is $1 \cdot \int_{\Sigma_0} f(z) d\mathcal{H}_0(\tau)$. Since in 0 dimensions the Hausdorff measure is the counting measure, the result follows. Now we assume the result for $n - 1$, i.e., for $w \in U^n$, we have

$$f^{[n-1]}(w) = \frac{1}{\sqrt{n}} \int_{\Sigma_{n-1}} f^{(n-1)}(\langle w, \tau \rangle) d\mathcal{H}_{n-1}(\tau).$$

Then,

$$f^{[n]}(z_1, \dots, z_{n+1}) = \frac{1}{z_n - z_{n+1}} \left(f^{[n-1]}(z_1, \dots, z_n) - f^{[n-1]}(z_1, \dots, z_{n-1}, z_{n+1}) \right) \quad (4.5)$$

$$\begin{aligned} &= \frac{1}{z_n - z_{n+1}} \frac{1}{\sqrt{n}} \int_{\Sigma_{n-1}} \left(f^{(n-1)}(\tau_1 z_1 + \dots + \tau_n z_n) \right. \\ &\quad \left. - f^{(n-1)}(\tau_1 z_1 + \dots + \tau_{n-1} z_{n-1} + \tau_n z_{n+1}) \right) d\mathcal{H}_{n-1}(\tau) \\ &= \frac{1}{\sqrt{n}} \int_{\Sigma_{n-1}} \tau_n \left(\int_0^1 f^{(n)} \left(\sum_{j=1}^{n-1} \tau_j z_j + s \tau_n z_n + (1-s) \tau_n z_{n+1} \right) ds \right) d\mathcal{H}_{n-1}(\tau) \end{aligned} \quad (4.6)$$

$$= \int_{\mathbf{A}_{n-1}} \tau_n \left(\int_0^1 f^{(n)} \left(\sum_{j=1}^{n-1} \tau_j z_j + s \tau_n z_n + (1-s) \tau_n z_{n+1} \right) ds \right) d\tau_1 d\tau_2 \dots d\tau_{n-1}. \quad (4.7)$$

To obtain the representation (4.5) from the recursive definition (4.1) we first switch z_1 with z_n , followed by a use of Corollary 4.2 to reorder the variables in the symmetric function $f^{[n-1]}$. To obtain (4.6) we have used the formula

$$g(w_1) - g(w_2) = (w_1 - w_2) \int_0^1 g'(sw_1 + (1-s)w_2) ds,$$

which is justified since U is convex and therefore the line segment joining the points $w_1, w_2 \in U$ is also in U . In (4.7) we used the relation (4.3) and the fact that the point (τ_1, \dots, τ_n) ranges over Σ_n as the coordinates $(\tau_1, \dots, \tau_{n-1})$ range over \mathbf{A}_{n-1} .

Let $\theta_1, \dots, \theta_{n+1}$ denote the restrictions of the natural coordinates of \mathbb{R}^{n+1} to Σ_n . Therefore $(\theta_1, \dots, \theta_n)$ are standard coordinates on Σ_n and these coordinates range over the simplex \mathbf{A}_n when $(\theta_1, \dots, \theta_n) \in \Sigma_n$. We define a map

$$\Phi : [0, 1] \times \Sigma_{n-1} \rightarrow \Sigma_n$$

as follows: using the coordinates $(\tau_1, \dots, \tau_{n-1}) \in \mathbf{A}_{n-1}$ on Σ_{n-1} and the coordinates $(\theta_1, \dots, \theta_n) \in \mathbf{A}_n$ on Σ_n , the map is given as

$$[0, 1] \times \mathbf{A}_{n-1} \rightarrow \mathbf{A}_n \text{ with } (s, \tau_1, \dots, \tau_{n-1}) \mapsto (\theta_1, \dots, \theta_n)$$

where

$$\theta_j = \tau_j \text{ for } j = 1, \dots, n-1, \text{ and } \theta_n = s \left(1 - \sum_{j=1}^{n-1} \tau_j \right).$$

A computation shows that the absolute value of the Jacobian determinant is given by

$$|\det \Phi'(s, \tau)| = 1 - \sum_{j=1}^{n-1} \tau_j = \tau_n.$$

Therefore, (4.7) can be written as

$$\begin{aligned} f^{[n]}(z) &= \int_{[0,1] \times \mathbf{A}_{n-1}} f^{(n)}(\langle z, \Phi(s, \tau) \rangle) |\det \Phi'(s, \tau)| ds d\tau_1 \dots d\tau_{n-1} \\ &= \int_{\mathbf{A}_n} f(\langle z, \theta \rangle) d\theta_1 \dots d\theta_n, \end{aligned}$$

where in the last line we use the change of variables formula. Finally using (4.3) (for $d = n + 1$) we conclude that

$$f^{[n]}(z) = \frac{1}{\sqrt{n+1}} \int_{\Sigma_n} f^{(n)}(\langle z, \theta \rangle) d\mathcal{H}_n(\theta).$$

The result now follows by induction. □

5. REGULARITY OF DIVIDED DIFFERENCES

5.1. Notation. Recall that $\mathcal{C}^{k,\alpha}(\Gamma)$ is the space of functions on the smooth curve Γ which are k times continuously differentiable with respect to arc length, and such that the k -th derivative is Hölder continuous of order α . This linear space becomes a Banach space with the norm

$$\|\phi\|_{\mathcal{C}^{k,\alpha}(\Gamma)} = \|\phi\|_{\mathcal{C}^k(\Gamma)} + |\phi^{(k)}|_\alpha,$$

with $\|\phi\|_{\mathcal{C}^k} = \sum_{j=0}^k \|\phi^{(j)}\|_\infty$, where $\|\cdot\|_\infty$ is the sup norm and the derivatives are taken with respect to arc length on Γ , and $|\cdot|_\alpha$ is the Hölder semi-norm defined as

$$|\psi|_\alpha = \sup_{s \neq s'} \frac{|\psi(s) - \psi(s')|}{|s - s'|^\alpha}.$$

In dealing with functions of several variables we use the standard multi-index conventions. Recall that $\mathcal{A}^{k,\alpha}(\Omega)$ denotes the space $\mathcal{C}^{k,\alpha}(\Omega) \cap \mathcal{O}(\Omega)$. For a $\Phi \in \mathcal{A}^{k,\alpha}(\Omega)$, where $\Omega \subseteq \mathbb{C}^n$ is a bounded domain, we use the standard norm

$$\|\Phi\|_{\mathcal{C}^{k,\alpha}(\Omega)} = \|\Phi\|_{\mathcal{C}^k(\overline{\Omega})} + \sum_{|\gamma|=k} |\partial^\gamma \Phi|_\alpha,$$

where

$$\partial^\gamma = \left(\frac{\partial}{\partial z_1} \right)^{\gamma_1} \dots \left(\frac{\partial}{\partial z_n} \right)^{\gamma_n} \text{ and } \|\Phi\|_{\mathcal{C}^k(\overline{\Omega})} = \sum_{|\gamma| \leq k} \|\partial^\gamma \Phi\|_\infty.$$

Also, $|\cdot|_\alpha$ denotes the Hölder α -seminorm of a function given by

$$|\Psi|_\alpha = \sup_{z \neq w} \frac{|\Psi(z) - \Psi(w)|}{|z - w|^\alpha}.$$

5.2. Hölder regularity of divided differences. Recall that Δ^{n-1} denotes the map which sends a holomorphic function f to its $(n-1)$ -th divided difference $f^{[n-1]}$.

Lemma 5.1. *Let U be a convex domain in \mathbb{C} . Then, Δ^{n-1} is continuous from $\mathcal{C}^{k+n-1,\alpha}(U)$ to $\mathcal{C}^{k,\alpha}(U^n)$.*

Proof. Thanks to the fact that $\mathcal{T}(f|_{\partial U}) = f$ for a function in the space $\mathcal{A}(U) = \mathcal{O}(U) \cap \mathcal{C}(\overline{U})$ of holomorphic functions continuous up to the boundary on U , we have from (4.2) that $\Delta^{n-1}f = \mathcal{B}_n f$ for $f \in \mathcal{C}^{k+n-1,\alpha}(U)$. Since \mathcal{B}_n is given by an integral, it is clearly a closed operator, so that by the closed graph theorem, it suffices to verify that for each $f \in \mathcal{C}^{k+n-1,\alpha}(U)$, we have $\mathcal{B}_n f \in \mathcal{C}^{k,\alpha}(U)$. Since U is convex, we can use the representation (4.4). For a multi-index $\gamma \in \mathbb{N}^n$, if $|\gamma| \leq k$, we can repeatedly differentiate under the integral sign to obtain

$$\partial^\gamma f^{[n-1]}(z) = \frac{1}{\sqrt{n}} \int_{\Sigma_{n-1}} \tau^\gamma \cdot f^{(|\gamma|+n-1)}(\langle z, \tau \rangle) d\mathcal{H}_{n-1}(\tau).$$

Noting that $\|\tau^\gamma\|_\infty \leq 1$ for $\tau \in \Sigma_{n-1}$, we have $\|\partial^\gamma f^{[n-1]}\|_\infty \leq C \|f^{(|\gamma|+n-1)}\|_\infty$. It further follows that if $|\gamma| < k$, then each derivative $\partial^\gamma f^{[n-1]}$ is uniformly continuous and therefore extends to $\overline{U^n}$ continuously. Consequently, $f^{[n-1]} \in \mathcal{C}^{k-1}(\overline{U^n})$. Now let $z, w \in U^n$ and let $\gamma \in \mathbb{N}^n$ be a multi-index such that $|\gamma| = k$. Then,

$$\begin{aligned} \left| \partial^\gamma f^{[n-1]}(z) - \partial^\gamma f^{[n-1]}(w) \right| &= \left| \frac{1}{\sqrt{n}} \int_{\Sigma_{n-1}} \tau^\gamma \cdot \left(f^{(|\gamma|+n-1)}(\langle z, \tau \rangle) - f^{(|\gamma|+n-1)}(\langle w, \tau \rangle) \right) d\mathcal{H}_{n-1}(\tau) \right| \\ &\leq \frac{1}{\sqrt{n}} \left| f^{(k+n-1)} \right|_\alpha \cdot \int_{\Sigma_{n-1}} \tau^\gamma |\langle z, \tau \rangle - \langle w, \tau \rangle|^\alpha d\mathcal{H}_{n-1}(\tau) \\ &= \frac{1}{\sqrt{n}} \left| f^{(k+n-1)} \right|_\alpha \cdot \int_{\Sigma_{n-1}} \tau^\gamma |\langle z - w, \tau \rangle|^\alpha d\mathcal{H}_{n-1}(\tau) \\ &\leq \frac{1}{\sqrt{n}} \left| f^{(k+n-1)} \right|_\alpha \cdot \int_{\Sigma_{n-1}} \tau^\gamma |\tau|^\alpha |z - w|^\alpha d\mathcal{H}_{n-1}(\tau) \\ &= C \cdot \left| f^{(k+n-1)} \right|_\alpha \cdot |z - w|^\alpha. \end{aligned} \tag{5.1}$$

It follows that for $|\gamma| = k$, the function $\partial^\gamma f^{[n-1]}$ is Hölder continuous and therefore uniformly continuous on U^n , and therefore extends continuously to $\overline{U^n}$, so that $f^{[n-1]} \in \mathcal{C}^k(\overline{U^n})$. Combining with the Hölder continuity of the k -th partials again, we see that $f^{[n-1]} \in \mathcal{C}^{k,\alpha}(U^n)$. The proof is complete. \square

6. PROOF OF THEOREM 1.3

6.1. The case $n = 1$. When $n = 1$, Theorem 1.3 is a well known classical result (see [11, 13, 14].)

6.2. U convex, and $n \geq 2$. In this case the result follows from the representation (4.2) and Lemma 5.1. Indeed $\mathcal{B}_n = \Delta^{n-1} \circ \mathcal{T}$, and since we know that $\mathcal{T} : \mathcal{C}^{k+n-1,\alpha}(\Gamma) \rightarrow \mathcal{C}^{k+n-1,\alpha}(U)$ and $\Delta^{n-1} : \mathcal{C}^{k+n-1,\alpha}(U) \rightarrow \mathcal{C}^{k,\alpha}(U^n)$ are continuous, the result follows.

6.3. U simply connected. Suppose that $U \subset \mathbb{C}$ is a nonconvex, simply connected domain. Let $\phi \in \mathcal{C}^{k+n-1,\alpha}(\Gamma)$, where $\Gamma = \partial U$. Let $f = \mathcal{T}\phi \in \mathcal{C}^{k+n-1,\alpha}(U)$. By the Riemann mapping theorem there exists a biholomorphism $\Psi : \mathbb{D} \rightarrow U$. The boundary regularity of Ψ may be deduced from the classical *Kellogg-Warschawski Theorem* which we will now recall.

Result 6.1 (See [19]). *Let $f : G \rightarrow D$ be a proper holomorphic mapping of planar domains. If for some integer $k \geq 1$ the boundaries of G and D are of class \mathcal{C}^k , then the mapping f is of class $\mathcal{C}^{k-1,\theta}(G)$, for each $0 < \theta < 1$.*

(*Remark:* In the classical literature, in this and other results on boundary regularity of holomorphic maps in one complex variable, it is usually assumed that f is the Riemann map from a simply connected domain to the unit disc. However, since the proofs depend only on local considerations at the boundary, these extend immediately to the more general situation of proper holomorphic mappings of domains. Recall that in one variable and for bounded domains, a proper holomorphic map is a local biholomorphism off a finite set of branch points.)

Since Γ is of class \mathcal{C}^{k+n+1} , it follows that for each $0 < \theta < 1$ we have $\Psi \in \mathcal{C}^{k+n,\theta}(\mathbb{D})$. Let $\Lambda = \Psi^{-1}$, then by another application of the same result, $\Lambda \in \mathcal{C}^{k+n,\theta}(U)$. Let $g = f \circ \Psi$, so that $f = g \circ \Lambda$. We first wish to show that $g \in \mathcal{C}^{k+n-1,\alpha}(\mathbb{D})$.

We denote by $\Psi_j(z)$ the jet of order j of the function Ψ at the point z :

$$\Psi_j(z) = (\Psi(z), \Psi'(z), \dots, \Psi^{(j)}(z)).$$

Recall the Faà di Bruno formula, which says that we may express the ℓ -th derivative of g as

$$g^{(\ell)}(z) = \sum_{j=1}^{\ell} f^{(j)}(\Psi(z)) B_j(\Psi_j(z)), \quad (6.1)$$

where $B_j : \mathbb{C}^{\ell+1} \rightarrow \mathbb{C}$ are holomorphic polynomials. (Although we do not need this information, the B_j 's are given explicitly as

$$B_j(p_0, p_1, \dots, p_\ell) = \sum \frac{\ell!}{b_1! \dots b_\ell!} \left(\frac{p_1}{1!}\right)^{b_1} \dots \left(\frac{p_\ell}{\ell!}\right)^{b_\ell},$$

where the sum is over all solutions in nonnegative integers b_k to the equations $b_1 + 2b_2 + \dots + \ell b_\ell = \ell$ and $b_1 + \dots + b_\ell = j$.) If we introduce the function F_j given by $F_j(p_0, \dots, p_\ell) = f^{(j)}(p_0) B_j(p_1, \dots, p_\ell)$, we can rewrite (6.1) as

$$g^{(\ell)}(z) = \sum_{j=1}^{\ell} F_j(\Psi_\ell(z)).$$

Note that $F_j \in \mathcal{C}^\alpha(U^\ell)$ if $j \leq k+n-1$, because F_j is the product of Hölder continuous functions of order α . Then, we have,

$$\begin{aligned} \left| g^{(k+n-1)}(z) - g^{(k+n-1)}(w) \right| &= \left| \sum_{j=1}^{k+n-1} F_j(\Psi_{k+n-1}(z)) - F_j(\Psi_{k+n-1}(w)) \right| \\ &\leq \sum_{j=1}^{k+n-1} |F_j(\Psi_{k+n-1}(z)) - F_j(\Psi_{k+n-1}(w))| \\ &\leq \sum_{j=1}^{k+n-1} |F_j|_\alpha |\Psi_{k+n-1}(z) - \Psi_{k+n-1}(w)|^\alpha \quad (\text{using the Hölder condition}) \\ &= C |\Psi_{k+n-1}(z) - \Psi_{k+n-1}(w)|^\alpha. \\ &\leq C |z - w|^\alpha, \end{aligned}$$

where the last step holds since $\Psi_{k+n-1} \in \mathcal{C}^{1,\theta}(\mathbb{D})$ for each $0 < \theta < 1$. It follows therefore that $g \in \mathcal{C}^{k+n-1,\alpha}(\mathbb{D})$.

Recall that $f^{[n-1]} = (g \circ \Lambda)^{[n-1]}$ where $\Lambda = \Psi^{-1}$ is the Riemann map from U to \mathbb{D} , and Λ is in $\mathcal{C}^{k+n,\theta}(U)$ for $0 < \theta < 1$. We use an analog of the Faà di Bruno formula for divided differences.

By [10, Theorem 1], at a point $z \in U^n$ the $(n-1)$ -th divided difference of the composite function $f = g \circ \Lambda$ is given by

$$f^{[n-1]}(z) = \sum_{j=1}^{n-1} \Lambda^{\{j, n-1\}}(z) \cdot g^{[j]}(\Lambda_j(z)), \quad (6.2)$$

where we have $\Lambda_j(z) = (\Lambda(z_1), \Lambda(z_2), \dots, \Lambda(z_{j+1}))$ (a discrete analog of the j -jet), and the $\Lambda^{\{j, n-1\}}$ are the functions on U^n given by

$$\Lambda^{\{j, n-1\}}(z) = \sum_{j=1}^{n-1} \left(\Lambda^{[1]}(z_1, \cdot) \Lambda^{[1]}(z_2, \cdot) \dots \Lambda^{[1]}(z_j, \cdot) \right)^{[n-j-1]}(z_{j+1}, \dots, z_n).$$

Note that $\Lambda \in \mathcal{A}^{k+n, \theta}(U)$, we claim that:

$$\Lambda^{\{j, n-1\}} \in \mathcal{A}^{k+j, \theta}(U^n). \quad (6.3)$$

Since $\Lambda^{\{j, n-1\}}$ is a polynomial in divided differences of Λ of order at most $(n-j)$, the claim would follow if we can show that $\Lambda^{[\ell]} \in \mathcal{A}^{k+n-\ell, \theta}(U^{\ell+1})$. This would follow from Lemma 5.1 if U were convex, but we will instead use the fact that Λ is a biholomorphism, and its inverse Ψ is indeed defined on the convex domain \mathbb{D} , and therefore Lemma 5.1 does apply to it. By repeatedly taking divided differences, we see that

$$\Lambda^{[\ell]}(z_1, \dots, z_{\ell+1}) = \frac{P}{\prod_{i < j} \Psi^{[1]}(\Lambda(z_i), \Lambda(z_j))},$$

where P is a polynomial in divided differences of Ψ of order at most ℓ , evaluated at the points $\Lambda(z_1), \dots, \Lambda(z_{\ell+1})$. Note that by injectivity of Ψ , the denominator is a nonvanishing function on $U^{\ell+1}$ of class $\mathcal{A}^{k+n-\ell, \theta}(U^{\ell+1})$. Claim (6.3) follows, since, products of Hölder functions and reciprocals of nonvanishing Hölder functions are Hölder of the same class.

Since $g \in \mathcal{O}(\mathbb{D})$, and \mathbb{D} is convex, we can represent the function $g^{[j]}$ using the Genocchi-Hermite formula. Putting this into (6.2), we obtain

$$f^{[n-1]}(z) = \sum_{j=1}^{n-1} \frac{\Lambda^{\{j, n-1\}}(z)}{\sqrt{j+1}} \int_{\Sigma_j} g^{(j)}(\langle \Lambda_j(z), \tau \rangle) d\mathcal{H}_j(\tau)$$

Let γ be a multi-index with $|\gamma| = k$. Differentiating under the integral sign and using the Leibniz rule for partial derivatives,

$$\begin{aligned} \partial^\gamma f^{[n-1]}(z) &= \sum_{j=1}^{n-1} \frac{1}{\sqrt{j+1}} \sum_{0 \leq \beta \leq \gamma} \binom{\gamma}{\beta} \partial^{\gamma-\beta} \Lambda^{\{j, n-1\}}(z) \cdot \partial^\beta \left(\int_{\Sigma_j} g^{(j)}(\langle \Lambda_j(z), \tau \rangle) d\mathcal{H}_j(\tau) \right) \\ &= \sum_{j=1}^{n-1} \frac{1}{\sqrt{j+1}} \sum_{0 \leq \beta \leq \gamma} \binom{\gamma}{\beta} \partial^{\gamma-\beta} \Lambda^{\{j, n-1\}}(z) \left(\int_{\Sigma_j} g^{(j+|\beta|)}(\langle \Lambda_j(z), \tau \rangle) \cdot \tau^\beta \cdot (\Lambda'_{n-1}(z))^\beta d\mathcal{H}_j(\tau) \right), \end{aligned}$$

where $\Lambda'_{n-1}(z) = (\Lambda'(z_1), \dots, \Lambda'(z_n)) \in \mathbb{C}^n$. If we define for $0 \leq \ell \leq k$ and $z \in U^n$,

$$\mathbf{v}_\ell(z) = \sum_{\substack{0 \leq \beta \leq \gamma \\ |\beta| = \ell}} \binom{\gamma}{\beta} \partial^{\gamma-\beta} \Lambda^{\{j, n-1\}}(z) \cdot (\Lambda'_{n-1}(z))^\beta,$$

then by (6.3), $\partial^{\gamma-\beta} \Lambda^{\{j,n-1\}} \in \mathcal{A}^{|\beta|,\theta}(U^n)$, and since Λ is in $\mathcal{C}^{k+n,\theta}(U)$, we conclude that $\Lambda'_{n-1} \in \mathcal{A}^{k+n-1,\theta}(U^n)$. It follows that for each $0 \leq \ell \leq k$ and for each $0 < \theta < 1$, we have $V_\ell \in \mathcal{A}^\theta(U^n)$. In terms of the V_ℓ , we can write

$$\partial^\gamma f^{[n-1]}(z) = \sum_{j=1}^{n-1} \sum_{\ell=1}^k \frac{1}{\sqrt{j+1}} V_\ell(z) \cdot \int_{\Sigma_j} g^{(j+\ell)}(\langle \Lambda_j(z), \tau \rangle) h_\ell^{j+1}(\tau) d\mathcal{H}_j(\tau), \quad (6.4)$$

where $h_\ell^{j+1}(\tau) = \sum_{|\beta|=\ell} \tau^\beta$ is the complete symmetric polynomial in $(j+1)$ variables of degree ℓ . Now

$$\begin{aligned} & \left| \int_{\Sigma_j} g^{(j+\ell)}(\langle \Lambda_j(z), \tau \rangle) h_\ell^{j+1}(\tau) d\mathcal{H}_j(\tau) - \int_{\Sigma_j} g^{(j+\ell)}(\langle \Lambda_j(w), \tau \rangle) h_\ell^{j+1}(\tau) d\mathcal{H}_j(\tau) \right| \\ &= \left| \int_{\Sigma_j} \left(g^{(j+\ell)}(\langle \Lambda_j(z), \tau \rangle) - g^{(j+\ell)}(\langle \Lambda_j(w), \tau \rangle) \right) h_\ell^{j+1}(\tau) d\mathcal{H}_j(\tau) \right| \\ &\leq C \left| \int_{\Sigma_j} |\langle \Lambda_j(z), \tau \rangle - \langle \Lambda_j(w), \tau \rangle|^\alpha h_\ell^{j+1}(\tau) d\mathcal{H}_j(\tau) \right| \\ &\leq C \left| \int_{\Sigma_j} |\langle \Lambda_j(z) - \Lambda_j(w), \tau \rangle|^\alpha h_\ell^{j+1}(\tau) d\mathcal{H}_j(\tau) \right| \\ &\leq C \left| \int_{\Sigma_j} |\tau|^\alpha |\Lambda_j(z) - \Lambda_j(w)|^\alpha h_\ell^{j+1}(\tau) d\mathcal{H}_j(\tau) \right| \\ &\leq C |\Lambda_j(z) - \Lambda_j(w)|^\alpha \\ &\leq C |z - w|^\alpha. \end{aligned}$$

Therefore the integral factor in each term of the double sum in (6.4) is a Hölder continuous function of order α , and since V_ℓ is also Hölder continuous of order α , we see that $\partial^\gamma f^{[n-1]} \in \mathcal{A}^\alpha(U^n)$. It now follows that $f^{[n-1]} = \mathcal{B}_n \phi \in \mathcal{C}^{k,\alpha}(U^n)$.

6.4. Multiply Connected Case. Now, let U be a domain with h holes, that is, U is $(h+1)$ -connected. We claim that *we can find $h+1$ simply connected domains $\{U_j\}_{j=1}^{h+1}$, each with boundary of class \mathcal{C}^{n+k+1} and an $R > 0$, such that*

- (1) $\bigcup_{j=1}^{h+1} U_j = U$, and
- (2) if $z, w \in U$ are such that $|z - w| < R$, for at least one j we have $z, w \in U_j$.

Assuming the claim, let as before $f = \mathcal{B}_1 \phi$. By the previous argument for the simply-connected case, we see that $(f|_{U_j})^{[n-1]} \in \mathcal{C}^{k,\alpha}(U_j)$. Therefore, in particular, $f^{[n-1]}$ is bounded on $\overline{U^n}$. Further, if $|z - w| < R$, we see that for any multi-index γ with $|\gamma| = k$, we have that $|\partial^\gamma f^{[n-1]}(z) - \partial^\gamma f^{[n-1]}(w)| \leq C |z - w|^\alpha$, since such z, w would belong to a common U_j by the claim. Therefore, applying [13, Lemma 7.3], we conclude that $f^{[n-1]} = \mathcal{B}_n \phi \in \mathcal{C}^{k,\alpha}(U^n)$. This completes the proof of Theorem 1.3, provided we justify the claim made above.

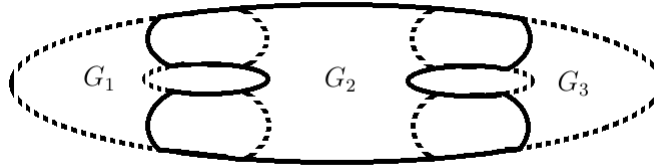
6.5. Proof of Claim. Given two subsets V, W of the plane, we say that they are *well-separated* if we have

$$\overline{V \setminus W} \cap \overline{W \setminus V} = \emptyset.$$

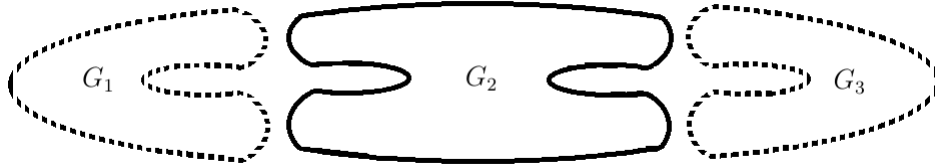
It is not difficult to see that Condition (2) of the claim is equivalent to demanding that each pair of simply-connected sets U_j and U_ℓ in the collection $\{U_j\}$ is well-separated. Note further that the condition of well-separatedness is invariant under homeomorphisms, which we will take advantage of by replacing the domain U with homeomorphic domains with simpler geometry. Indeed if $\Psi : \bar{U} \rightarrow \bar{G}$ is a diffeomorphism of class \mathcal{C}^{k+n+1} then it suffices to find a cover of G by simply connected open sets $\{G_i\}_{i=1}^{h+1}$, each with \mathcal{C}^{k+n+1} boundary, and such that each pair of sets G_i and G_j is well-separated. We will illustrate the fact that this is always possible by using pictures. We may take G to look like in the picture below (in this case we take $h = 2$):



And it is clear how to construct the sets G_1, G_2, G_3 in such a way that they have \mathcal{C}^{n+k+1} boundary and are pairwise well-separated. In the picture below, we have drawn the boundaries of G_1 and G_3 in dotted lines and that of G_2 in bold lines to illustrate the idea:



The following picture shows the three domains G_1, G_2, G_3 separately:



7. PROOF OF THEOREM 1.2

For $z, w \in \mathbb{C}^n$, define

$$\delta(z, w) = \min_{\sigma \in S_n} |z - \sigma(w)|,$$

where $\sigma(w)$ is as in (1.11). This may be thought of as a natural metric on the quotient \mathbb{C}^n/S_n . We compare the distance induced in this way on the symmetric product U^n/S_n with the metric on the realization $\pi(U^n) = \Sigma^n U$. We will use the global Łojasiewicz inequality of Ji-Kollár-Shiffman (see [12, Corollary 6, Equation 6.2]), which states the following. Let $f_1, \dots, f_n \in \mathbb{C}[z_1, \dots, z_N]$ and for $1 \leq i \leq n$, let $d_i = \deg f_i$. Let $Z \subset \mathbb{C}^N$ be the common zero set of the polynomials $\{f_i\}_{i=1}^n$. Then there is a constant $C > 0$ such that

$$\left(\frac{\text{dist}(z, Z)}{1 + |z|^2} \right)^{\bar{B}(N, d_1, \dots, d_n)} \leq C \cdot \max_i |f_i(z)|, \quad (7.1)$$

where the constant $\bar{B}(N, d_1, \dots, d_n)$ can be computed explicitly (see [12, Section 3].)

Lemma 7.1. *For any domain $U \subset \mathbb{C}$, there is a $C > 0$, such that for $z, w \in U^n$, we have*

$$\delta(z, w)^{\Lambda_n} \leq C |\pi(z) - \pi(w)|,$$

where Λ_n is as in (1.3).

Proof. We will use (7.1) when $N = 2n$ with coordinates (z, w) on $\mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$, where $z, w \in \mathbb{C}^n$. For $1 \leq i \leq n$, we set $f_i(z, w) = \pi_i(z) - \pi_i(w)$, so that $d_i = i$. In the notation of Section 3 of [12], we have

$$\overline{B}(2n, d_1, \dots, d_n) = \left(\frac{3}{2}\right)^j B(2n, d_1, \dots, d_n) + \theta$$

Since $d_i = i$ for $i \leq n$, we have $B(2n, d_1, \dots, d_n) = n!$. We also have that $\theta = 0$ since the dimension of the space is $2n$ and the number of equations is $k = n$. Then $j = \#\{i < n - 1 \mid i = 2\}$, so that $j = 0$ if $n \leq 3$ and $j = 1$ otherwise. Thus

$$\overline{B}(2n, d_1, \dots, d_n) = \Lambda_n = \begin{cases} n! & \text{if } n \leq 3 \\ \frac{3}{2}n! & \text{if } n > 3. \end{cases}$$

Therefore, we conclude that

$$\left(\frac{\text{dist}((z, w), Z)}{1 + |z|^2 + |w|^2}\right)^{\Lambda_n} \leq C \cdot \max_i |\pi_i(z) - \pi_i(w)|, \quad (7.2)$$

where $Z = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n \mid \pi_i(z) - \pi_i(w) = 0 \text{ for } 1 \leq i \leq n\}$. Clearly if $(z, w) \in Z$ then for some $\sigma \in S_n$ we have $z = \sigma(w)$. Let us denote for $\sigma \in S_n$

$$L_\sigma = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n \mid z = \sigma(w)\},$$

which is an n -dimensional linear subspace of the complex vector space \mathbb{C}^{2n} . We can alternatively represent L_σ as:

$$\begin{aligned} L_\sigma &= \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n \mid z_j - w_{\sigma(j)} = 0 \text{ for } 1 \leq j \leq n\} \\ &= \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n \mid \langle (z, w), e_j - e_{\sigma(j)+n} \rangle = 0, 1 \leq j \leq n\} \end{aligned} \quad (7.3)$$

where $\{e_j\}_{j=1}^{2n}$ denotes the standard complex basis of \mathbb{C}^{2n} . We may write the affine algebraic variety Z as the union of irreducible algebraic subvarieties as follows:

$$Z = \bigcup_{\sigma \in S_n} L_\sigma \quad (7.4)$$

Then it is clear by (7.3) that the orthogonal complement to L_σ (in $\mathbb{C}^n \times \mathbb{C}^n$) is given by

$$L_\sigma^\perp = \text{span} \left\{ \frac{1}{\sqrt{2}}(e_j - e_{\sigma(j)+n}), 1 \leq j \leq n \right\}.$$

The length of the projection of a point $(z, \sigma(w))$ onto L_σ^\perp is given by:

$$\begin{aligned}
|P_{L_\sigma^\perp}(z, w)| &= \left| \sum_{j=1}^n \left\langle (z, w), \frac{1}{\sqrt{2}}(e_j - e_{\sigma(j)+n}) \right\rangle \frac{1}{\sqrt{2}}(e_j - e_{\sigma(j)+n}) \right| \\
&= \frac{1}{2} \left| \sum_{j=1}^n (z_j - w_{\sigma(j)})(e_j - e_{\sigma(j)+n}) \right| \\
&= \frac{1}{2} \left(\sum_{j=1}^n 2 |z_j - w_{\sigma(j)}|^2 \right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}} |z - \sigma(w)|.
\end{aligned}$$

Using (7.4):

$$\begin{aligned}
\text{dist}((z, w), Z) &= \min_{\sigma \in S_n} \text{dist}((z, w), L_\sigma) \\
&= \min_{\sigma \in S_n} |P_{L_\sigma^\perp}(z, w)| \\
&= \min_{\sigma \in S_n} \frac{1}{\sqrt{2}} |z - \sigma(w)| \\
&= \frac{1}{\sqrt{2}} \delta(z, w).
\end{aligned}$$

Substituting the last equality in 7.2 we obtain:

$$\left(\frac{\frac{1}{\sqrt{2}} \delta(z, w)}{1 + |z|^2 + |w|^2} \right)^{\Lambda_n} \leq C \cdot \max_i |\pi_i(z) - \pi_i(w)|$$

Since the point (z, w) belongs to the bounded set $U^n \times U^n \subset \mathbb{C}^{2n}$ we have

$$\begin{aligned}
\delta(z, w)^{\Lambda_n} &\leq C \cdot \max_i |\pi_i(z) - \pi_i(w)| \\
&\leq C \cdot |\pi(z) - \pi(w)|,
\end{aligned}$$

which completes the proof. \square

7.1. Three operators. We introduce three operators π_* , M_γ and j_k^* which will be needed in the proof of Theorem 1.2. Let $\mathcal{O}_{\text{sym}}(U^n)$ be the space of symmetric holomorphic functions on U^n . Then we define a push-forward map $\pi_* : \mathcal{O}_{\text{sym}}(U^n) \rightarrow \mathcal{O}(\Sigma^n U)$ in the following way: for an $f \in \mathcal{O}_{\text{sym}}(U^n)$,

$$(\pi_* f)(z) = f(\zeta),$$

where $\zeta \in U^n$ is any point such that $\pi(\zeta) = z$.

Proposition 7.2. π_* is well-defined and maps $\mathcal{O}_{\text{sym}}(U^n)$ to $\mathcal{O}(\Sigma^n U)$.

Proof. If ζ and ζ' are such that $\pi(\zeta) = \pi(\zeta') = z$ then $\zeta' = \sigma(\zeta)$ for some $\sigma \in S_n$ (see Section 3). Therefore, $f(\zeta) = f(\zeta')$ as f is symmetric. This shows that the definition makes sense and $\pi_* f$ is a well defined function on $\Sigma^n U$.

Now we show that $\pi_* f$ is holomorphic. Off the analytic set

$$Z = \left\{ z \in U^n \mid \prod_{i < j} (z_i - z_j) = 0 \right\}$$

the map π is a local biholomorphism. It follows that $\pi_* f$ is holomorphic on $\pi(U^n \setminus Z) \supset \Sigma^n U \setminus \pi(Z)$, where $\pi(Z)$ is an analytic set by the Remmert Proper Mapping Theorem. Since $\pi_* f$ is bounded, by the Riemann Removable Singularity Theorem it extends holomorphically to $\Sigma^n U$. \square

For a multi-index $\gamma \in \mathbb{N}^n$, we define M_γ to be the operator which multiplies a function on Γ by the smooth function

$$u_\gamma(t) = (-1)^{|\gamma|} \cdot |\gamma|! \cdot t^{\sum_{j=1}^n \gamma_j (n-j)}, \quad (7.5)$$

so that

$$(M_\gamma \phi)(t) = u_\gamma(t) \phi(t).$$

For an integer $k \geq 0$, let j_k denote the diagonal embedding of \mathbb{C}^n in the k -fold product $\mathbb{C}^{nk} = (\mathbb{C}^n)^k = \mathbb{C}^n \times \cdots \times \mathbb{C}^n$, i.e., for a $z \in \mathbb{C}^n$, we have

$$j_k(z) = (z, \dots, z) \in \underbrace{\mathbb{C}^n \times \cdots \times \mathbb{C}^n}_{k \text{ times}}.$$

Let $\Omega \subset \mathbb{C}^n$, and let $\Omega^k = \Omega \times \cdots \times \Omega \subset \mathbb{C}^{nk}$. For a function f on Ω , we define the function $j_k^* f$ on Ω^k by

$$j_k^* f = f \circ j_k. \quad (7.6)$$

7.2. End of Proof of Theorem 1.2. A direct computation shows that

$$\partial^\gamma \left(\frac{1}{q_n(z, t)} \right) = \frac{u_\gamma(t)}{q_n(z, t)^{|\gamma|+1}},$$

Let $z \in \Sigma^n U$ and let $w \in U^n$ be such that $z = \pi(w)$. For a continuous ϕ on Γ , we have :

$$\begin{aligned} \partial^\gamma \mathcal{E}_n \phi(z) &= \frac{1}{2\pi i} \int_\Gamma \frac{M_\gamma \phi(t)}{q_n(z, t)^{|\gamma|+1}} dt \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{M_\gamma \phi(t)}{q_n(\pi(w), t)^{|\gamma|+1}} dt \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{M_\gamma \phi(t)}{((t - w_1) \cdots (t - w_n))^{|\gamma|+1}} dt \\ &= j_{|\gamma|+1}^* (\mathcal{B}_{n(|\gamma|+1)} (M_\gamma \phi)) (w). \end{aligned}$$

Since $z = \pi(w)$ it follows that

$$\partial^\gamma \circ \mathcal{E}_n = \pi_* \circ j_{|\gamma|+1}^* \circ \mathcal{B}_{n(|\gamma|+1)} \circ M_\gamma. \quad (7.7)$$

To prove Theorem 1.2, it suffices to show that for each multi-index γ with $|\gamma| = k$, the operator $\partial^\gamma \mathcal{E}_n$ maps the space $\mathcal{C}^{n(k+1)-1, \alpha}(\Gamma)$ continuously into $\mathcal{C}^{\frac{\alpha}{\Lambda_n}}(\Sigma^n)$. By (7.7), we have $\partial^\gamma \circ \mathcal{E}_n = \pi_* \circ j_{k+1}^* \circ \mathcal{B}_{n(k+1)} \circ M_\gamma$. The operator M_γ , being multiplication by the smooth function u_γ , maps $\mathcal{C}^{n(k+1)-1, \alpha}(\Gamma)$ continuously to itself. Since the boundary Γ is of class $\mathcal{C}^{n(k+1)+1}$, it follows by

Theorem 1.3, that the operator $\mathcal{B}_{n(k+1)}$ maps $\mathcal{C}^{n(k+1)-1,\alpha}(\Gamma)$ continuously to $\mathcal{C}^{0,\alpha}(U^{n(k+1)})$. Note that j_k^* , defined by (7.6) maps $\mathcal{C}^{0,\alpha}(U^{n(k+1)})$ to $\mathcal{C}^{0,\alpha}(U^n)$. Indeed,

$$\begin{aligned} |j_k^* f(z) - j_k^* f(w)| &= |f(j_k(z)) - f(j_k(w))| \\ &\leq |f|_\alpha |j_k(z) - j_k(w)|^\alpha \\ &= |f|_\alpha |j_k(z - w)|^\alpha \\ &= |f|_\alpha n^{\frac{\alpha}{2}} |z - w|^\alpha. \end{aligned}$$

Therefore, j_k^* maps $\mathcal{A}^\alpha(U^{n(k+1)})$ continuously to $\mathcal{A}^\alpha(U^n)$. To complete the proof it suffices to show that π_* maps $\mathcal{A}^\alpha(U^n)$ continuously to $\mathcal{A}^{\frac{\alpha}{\Lambda_n}}(\Sigma^n U)$. By Proposition 7.2, π_* preserves holomorphicity, so it suffices to show that π_* is continuous from $\mathcal{C}^\alpha(U^n)$ to $\mathcal{C}^{\frac{\alpha}{\Lambda_n}}(\Sigma^n U)$. If $\pi(\zeta) = z$ and $\pi(\lambda) = w$ we have

$$\begin{aligned} |\pi_* f(z) - \pi_* f(w)| &= |f(\zeta) - f(\lambda)| \\ &\leq |f|_\alpha |\zeta - \lambda|^\alpha \end{aligned}$$

Then, for any $\sigma \in S_n$, we know that $w = \pi(\sigma(\lambda))$. Therefore, the relation

$$|\pi_* f(z) - \pi_* f(w)| \leq |f|_\alpha |\zeta - \sigma(\lambda)|^\alpha$$

holds for every $\sigma \in S_n$. Hence

$$\begin{aligned} |\pi_* f(z) - \pi_* f(w)| &\leq |f|_\alpha \min_{\sigma \in S_n} |\zeta - \sigma(\lambda)|^\alpha \\ &= |f|_\alpha \delta(\zeta, \lambda)^\alpha \\ &\leq C \cdot |f|_\alpha |\pi(\zeta) - \pi(\lambda)|^{\frac{\alpha}{\Lambda_n}} && \text{By Lemma 7.1} \\ &= C \cdot |f|_\alpha |z - w|^{\frac{\alpha}{\Lambda_n}}. \end{aligned}$$

Thus

$$\begin{aligned} |\pi_* f|_{\frac{\alpha}{\Lambda_n}} &= \sup_{z \neq w} \frac{|\pi_* f(z) - \pi_* f(w)|}{|z - w|^{\frac{\alpha}{\Lambda_n}}} \\ &\leq C |f|_\alpha \end{aligned} \tag{7.8}$$

Also we have

$$\begin{aligned} \|\pi_* f\|_\infty &= \sup_{z \in \Sigma^n U} |\pi_* f(z)| \\ &= \sup_{z \in \Sigma^n U} |f(\pi^{-1}(z))| \\ &= \sup_{\zeta \in U^n} |f(\zeta)| \\ &= \|f\|_\infty \end{aligned} \tag{7.9}$$

Combining equations (7.8) and (7.9) we obtain

$$\begin{aligned} \|\pi_* f\|_{\mathcal{C}^{\frac{\alpha}{\Lambda_n}}(\Sigma^n U)} &= |\pi_* f|_{\frac{\alpha}{\Lambda_n}} + \|\pi_* f\|_\infty \\ &\leq C |f|_\alpha + \|f\|_\infty \\ &\leq C \|f\|_\alpha \end{aligned}$$

Thus, π_* maps $\mathcal{A}_{\text{sym}}^\alpha(U^n)$ continuously to $\mathcal{A}^{\frac{\alpha}{\Lambda_n}}(\Sigma^n U)$. The proof of Theorem 1.2 is complete.

8. RANGE OF THE OPERATORS \mathcal{B}_n AND \mathcal{E}_n

We note here that for $n \geq 2$, the operator \mathcal{B}_n (resp. \mathcal{E}_n) is not surjective from the space $\mathcal{C}^{k+n-1,\alpha}(\Gamma)$ to $\mathcal{A}_{\text{sym}}^{k,\alpha}(U^n)$ (resp. from the space $\mathcal{C}^{(k+1)n-1,\alpha}(\Gamma)$ to the space $\mathcal{A}^{k,\frac{\alpha}{\Lambda_n}}(\Sigma^n U)$.) Indeed, it is not difficult to characterize the range of the operators \mathcal{B}_n and \mathcal{E}_n . Suppose the domain U has m holes, and for $1 \leq i \leq m$, we fix a point a_i in the i -th hole. Let h_p^q denote the *complete symmetric polynomial* in q variables of degree p given by:

$$h_p^q(z) = \sum_{|\gamma|=p} z^\gamma,$$

where we set $h_p^q = 0$ if $p < 0$. Consider the collection of functions $\mathcal{F} \subset \mathcal{O}_{\text{sym}}(U^n)$, with members of the form

$$h_r^n(z), \text{ and } \frac{1}{(z_1 - a_i) \dots (z_n - a_i)} h_{r-1}^n \left(\frac{1}{z_1 - a_i}, \dots, \frac{1}{z_n - a_i} \right)$$

where $r \in \mathbb{N} = \{0, 1, \dots\}$ and $1 \leq i \leq m$. We also let $\mathcal{F}_* = \{\pi_*(f) \mid f \in \mathcal{F}\}$. Note that the linear span of the collection of functions $\left\{ z^r, \frac{1}{(z - a_i)^r} \mid r \in \mathbb{N} \text{ and } 1 \leq i \leq m \right\}$ is dense in $\mathcal{A}^{k+n-1,\alpha}(U)$ (a consequence of the Mergelyan approximation theorem). Recall the representations $\mathcal{B}_n = \Delta^{n-1} \circ \mathcal{T}$ and $\mathcal{E}_n = \pi_* \circ \mathcal{B}_n$ (special case of (7.7) for $\gamma = 0$). Using these representations with the facts that $\mathcal{B}_n(z^r) = h_r^n(z)$ and

$$\mathcal{B}_n \left(\frac{1}{(z - a_i)^r} \right) = \frac{1}{(z_1 - a_i) \dots (z_n - a_i)} h_{r-1}^n \left(\frac{1}{z_1 - a_i}, \dots, \frac{1}{z_n - a_i} \right),$$

we can show the following:

Proposition 8.1. *The linear span of \mathcal{F} is dense in the range of $\mathcal{B}_n : \mathcal{C}^{k+n-1,\alpha}(\Gamma) \rightarrow \mathcal{C}^{k,\alpha}(U^n)$, and that of \mathcal{F}_* is dense in the range of $\mathcal{E}_n : \mathcal{C}^{(k+1)n-1,\alpha}(\Gamma) \rightarrow \mathcal{A}^{k,\frac{\alpha}{\Lambda_n}}(\Sigma^n U)$.*

For example the function $z_1 z_2 \in \mathcal{O}_{\text{sym}}(U^2)$ is not in the range of \mathcal{B}_2 .

9. PROOF OF THEOREM 1.1

We begin by recalling the structure of proper holomorphic maps between symmetric products of domains (see [4, Corollary 3] and [8, 9]). Given a holomorphic map $f : U \rightarrow V$, there is a unique map $\Sigma^n f : \Sigma^n U \rightarrow \Sigma^n V$ such that for $z \in U^n$ we have

$$(\Sigma^n f)(\pi(z_1, \dots, z_n)) = \pi(f(z_1), \dots, f(z_n)),$$

where $\pi : U^n \rightarrow \Sigma^n U$ is the symmetrization map as in (1.1). The map $\Sigma^n f$ is called the *n -fold symmetric product* of the map f . For further properties of the symmetric products of maps, see [4]. In particular, we need the following integral representation of the symmetric power $\Sigma^n f$ of a map $f : U \rightarrow \mathbb{C}$, when f extends continuously to ∂U (see [4, Proposition 2.4]): There is a polynomial automorphism \mathfrak{P} of \mathbb{C}^n , such that we can write

$$\Sigma^n f = \mathfrak{P} \circ \Psi, \tag{9.1}$$

where $\Psi = (\Psi_1, \dots, \Psi_n) : \Sigma^n U \rightarrow \mathbb{C}^n$ is the map whose ℓ -th component is given by

$$\Psi_\ell(z) = \frac{1}{2\pi i} \int_{\partial U} (f(t))^\ell \cdot \frac{q'_n(z, t)}{q_n(z, t)} dt. \tag{9.2}$$

Here $q_n(z, t)$ is as in (1.7):

$$q_n(z, t) = t^n - z_1 t^{n-1} + \cdots + (-1)^n z_n = \sum_{j=0}^n (-1)^j z_j t^{n-j},$$

where we set $z_0 = 1$ and

$$q'_n(z, t) = \frac{\partial q_n}{\partial t}(z, t) = \sum_{j=0}^{n-1} (-1)^j (n-j) z_j t^{n-j-1}. \quad (9.3)$$

If we let $\partial U = \Gamma$, then plugging in (9.3) into (9.2) we obtain the representation

$$\Psi_\ell(z) = \sum_{j=0}^{n-1} (-1)^j (n-j) z_j \mathcal{E}_n(t^{n-j-1} f|_{\partial U}) \quad (9.4)$$

Using the classical technique of Remmert and Stein (see [17, 15]), one can show that if U and V are bounded planar domains, then each proper holomorphic map $F : \Sigma^n U \rightarrow \Sigma^n V$ is of the form $F = \Sigma^n f$ for a proper holomorphic map $f : U \rightarrow V$ (see [8, 9, 4].) By hypothesis the boundaries of each of U and V is of class \mathcal{C}^{nk+n+1} . By the Kellogg-Warschawski theorem, (see Result 6.1 above), the proper holomorphic map $f : U \rightarrow V$ is of class $\mathcal{C}^{nk+n, \theta}(U)$ for each $0 < \theta < 1$, therefore, a fortiori of class $\mathcal{C}^{n(k+1)-1, \theta}(U)$. Then the restriction $f|_{\partial U} \in \mathcal{C}^{n(k+1)-1, \theta}(\partial U)$. Applying Theorem 1.2, each term in the sum (9.4) belongs to $\mathcal{A}^{k, \frac{\theta}{\Lambda_n}}(\Sigma^n U)$. Therefore the map $\Psi \in \mathcal{A}^{k, \frac{\theta}{\Lambda_n}}(\Sigma^n U)$, and since $F = \Sigma^n f = \mathfrak{P} \circ \Psi$, where \mathfrak{P} is a polynomial automorphism of \mathbb{C}^n , we have that $F \in \mathcal{A}^{k, \frac{\theta}{\Lambda_n}}(\Sigma^n U)$. The proof is complete.

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